Math. Proc. Camb. Phil. Soc. (2008), **144**, 267 © 2008 *Cambridge Philosophical Society* 267
doi:10.1017/S0305004107000862 *Printed in the United Kingdom*
First published online 26 February 2008

Asymptotic density in quasi-logarithmic additive number systems

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(*Received* 7 *September* 2006; *revised* 12 *December* 2006)

Abstract

We show that in quasi-logarithmic additive number systems $\mathcal{A}$ all partition sets have asymptotic density, and we obtain a corresponding monadic second-order limit law for adequate classes of relational structures. Our conditions on the local counting function $p(n)$ of the set of irreducible elements of $\mathcal{A}$ allow situations which are not covered by the density theorems of Compton [**6**] and Woods [**15**]. We also give conditions on $p(n)$ which are sufficient to show the assumptions of Compton's result are satisfied, but which are not necessarily implied by those of Bell and Burris [**2**], Granovsky and Stark [**8**] or Stark [**14**].

1. *Introduction and overview*

1·1. *Monadic second-order limit laws*

A problem that is addressed in finite model theory is to determine the probability that a finite structure, chosen randomly from a class $\mathcal{K}$ of structures, satisfies a given sentence φ of some logic. In this paper, we examine this problem for classes $\mathcal{K}$ of finite relational structures and a monadic second-order logic, extending previous results of Bell and Burris [**2**], Granovsky and Stark [**8**] and Stark [**14**].

We briefly introduce the notion of a monadic second-order limit law. Burris [**4**, chapter 6] serves as main reference. For an introduction to logic and model theory we refer to Ebbinghaus et al. [**7**] and Chang and Keisler [**5**], respectively. To construct our monadic second-order logic and to introduce $\mathcal{K}$, we start with a finite purely relational language L, a finite set of relation symbols along with their arities. A finite L-*structure* $\mathbf{S}$ of size n is a pair $(S, \mathcal{J})$, where S is a finite set with n elements, the universe of $\mathbf{S}$, and where $\mathcal{J}$ is an assignment of the relation symbols of L to relations on S which preserves arity. Thus, $\mathbf{S}$ consists of an appropriate interpretation $\mathcal{J}$ of the relation symbols of L in S. Two L-structures $\mathbf{S}_1 = (S_1, \mathcal{J}_1)$ and $\mathbf{S}_2 = (S_2, \mathcal{J}_2)$ are isomorphic if there is a bijection $f: S_1 \to S_2$, such that for each relation R_1 of $\mathbf{S}_1$ and the corresponding relation R_2 of $\mathbf{S}_2$ we have $R_1(x_1, \ldots, x_k)$ if and only if $R_2(f(x_1), \ldots, f(x_k))$ for all $x_1, \ldots, x_k \in S_1$. A first-order logic with language L consists, in addition to the symbols of L, of logical symbols: parentheses, the connectives $\wedge$ (and) and $\neg$ (not), the quantifier $\forall$ (for all), a binary relation symbol $\equiv$ (identity), and first-order variables, which range over elements of structures. A first-order L-formula, that is, roughly speaking, a "meaningful" combination of symbols, is defined inductively using prescribed rules, starting with atomic formulas (cf. Ebbinghaus et al. [**7**, section 3]). A first-order

L-sentence is a formula where each variable is bound by a quantifier. For each sentence φ an L-structure will either satisfy or fail to satisfy φ. A *monadic second-order logic* with language L extends the first-order logic by introducing unary relation variables. These range over subsets of structures. Monadic second-order L-formulas are obtained by augmenting the inductive definition of the first-order formulas by introducing $U(v)$ as an atomic formula for any unary relation variable U and any first-order variable v, and by defining $(\forall U \varphi)$ to be a monadic second-order L-formula if φ is one.

Now fix a class $\mathcal{K}$ of finite L-structures. The notion of probability mentioned at the beginning of this introduction is defined as the limit as $n \to \infty$, if it exists, of the proportion of isomorphy types of structures of size n in $\mathcal{K}$ that satisfy φ among all isomorphy types of structures of size n in $\mathcal{K}$. If this limit exists for all monadic second-order L-sentences, the class $\mathcal{K}$ is said to have a *monadic second-order (local) limit law*.

In the context of a monadic second-order logic with a finite, purely relational language L, Compton [**6**] introduced a method of proving such limit laws for $\mathcal{K}$ simply by analyzing the growth rate of $a(n)$, the number of isomorphy types of structures in $\mathcal{K}$ of size n. His method relies on the notion of an *adequate* class of structures. A class $\mathcal{K}$ of L-structures is adequate

(i) if it is closed under disjoint union,
(ii) if elements of $\mathcal{K}$ can be decomposed uniquely, up to commutativity and associativity, into a disjoint union of $\mathcal{K}$-indecomposable structures,
(iii) if the L-structure with empty universe is contained in $\mathcal{K}$,
(iv) and if, up to isomorphism, $\mathcal{K}$ contains only finitely many structures of each size.

If $\mathcal{K}$ is adequate, the set of isomorphism types of structures in $\mathcal{K}$ can be endowed naturally with the structure of an additive number system $\mathcal{A}_{\mathcal{K}}$. For such a class $\mathcal{K}$, the task of proving the existence of a monadic second-order limit law is reduced to the proof the existence of asymptotic density of partition sets in $\mathcal{A}_{\mathcal{K}}$.

In the following subsection we briefly recall the basic definitions related to additive number systems that we use in the proofs in this paper, and that are needed to state Compton's results in their precise forms. For a comprehensive introduction to the theory of additive number systems and Compton's approach to logical limit laws, we refer to Burris [**4**].

1·2. *Additive number systems, densities and partition sets*

Let $(A, +, 0)$ be a free commutative monoid, $0 \in A$ being the neutral element for the operation $+$. An element of $A \setminus \{0\}$ is called *indecomposable element* or *component* if it can not be written as a sum of two non-zero elements from A. We denote the set of all indecomposable elements of A by P. Let $\|\cdot\| : A \to \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ be an additive norm on A. For each $B \subset A$ and each $n \in \mathbb{Z}_+$ we define $B(n) := \{\omega \in B : \|\omega\| = n\}$, the subset of all elements of B of norm n. This gives rise to a map $b: \mathbb{Z}_+ \to \mathbb{Z}_+ \cup \{\infty\}$, defined by $b(n) := |B(n)|$, the *(local) counting function* of B. The generating series of B is denoted by $\mathbf{B}(x) := \sum_{n=0}^{\infty} b(n)x^n$. The counting functions of A and P are denoted by $a(n)$ and $p(n)$, respectively. Note that $a(0) = 1$, since $\|\cdot\|$ is a norm and thus $\|\omega\| = 0$ implies that $\omega = 0$.

The tuple $\mathcal{A} := (A, +, 0, P, \|\cdot\|)$ is an *additive number system*, if $(A, +, 0)$ is a free commutative monoid, P the set of indecompsable elements of A, and $\|\cdot\|$ an additive norm on A, such that $A(n)$ is finite for all $n \in \mathbb{N} := \{1, 2, 3, \dots\}$. The *radius of convergence* ρ of $\mathcal{A}$ is the radius of convergence of the generating series $\mathbf{A}(x) := \sum_{n=0}^{\infty} a(n)x^n$ of A. If $P \neq \emptyset$, which we assume from now on, then $a(n) \geqslant 1$ holds true for infinitely many

n, and we have $0 \leqslant \rho \leqslant 1$ in this case. The number system $\mathcal{A}$ is *reduced* if $\gcd\{n \in \mathbb{N} : p(n) > 0\} = 1$, which is equivalent to the property that $a(n) > 0$ for all n large enough (cf. Burris [**4**, lemma 2·42 and theorem 2·52]).

For $B \subset A$, the limit

$$\delta(B) := \lim_{\substack{n\to\infty \\ a(n)\neq 0}} \frac{b(n)}{a(n)},$$

provided it exists, is the *(local) asymptotic density* of B. If $\rho > 0$, we define the *Dirichlet density* of $B \subset A$ as the limit

$$\partial(B) := \lim_{x\to\rho^-} \frac{\mathbf{B}(x)}{\mathbf{A}(x)},$$

again provided it exists. For every $B \subset A$ and every $m \in \mathbb{Z}_+$ we set

$$mB := \begin{cases} \{0\} & \text{if } m = 0, \\ \{b_1 + \cdots + b_m : b_j \in B\} & \text{if } m \geqslant 1, \end{cases}$$

and define

$$(\leqslant m)B := \bigcup_{j=0}^{m} jB \qquad \text{and} \qquad (\geqslant m)B := \bigcup_{j=m}^{\infty} jB.$$

A set $B \subset A$ is a *partition set* of $\mathcal{A}$ if there is a partition of P into non-empty pairwise disjoint sets $P_1, \ldots, P_k$, and if there are non-negative integers $m_1, \ldots, m_k$, such that

$$B = \gamma_1 P_1 + \cdots + \gamma_k P_k,$$

where γ_i is of the form m_i, $(\leqslant m_i)$ or $(\geqslant m_i)$ for all $1 \leqslant i \leqslant k$. With these definitions, we can state Compton's results. The first is referred to as *Compton's density theorem*.

THEOREM 1·1 (Compton [**6**]). *Let $\mathcal{A}$ be a reduced additive number system with radius of convergence $0 < \rho < 1$. If there are constants $K > 0$ and $C > 0$, such that*

$$\frac{a(n-k)}{a(n)} \leqslant C\rho^k \qquad \textit{for all } (k, n) \textit{ with } K \leqslant k \leqslant n, \tag{1·1}$$

then all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density.

THEOREM 1·2 (Compton [**6**]). *Let L be a finite, purely relational language, and let $\mathcal{K}$ be an adequate class of finite L-structures. If all the partition sets of the additive number system $\mathcal{A}_{\mathcal{K}}$ have asymptotic density, then $\mathcal{K}$ has a local monadic second-order limit law.*

Woods's density theorem, a modification of Compton's first result, is used to prove monadic second-order limit laws for classes of unary functions with additional unary predicates.

THEOREM 1·3 (Woods [**15**]). *If there are constants $c > 0$, $0 < x < 1$ and $-\infty < \mu < 1$, such that*

$$p(n) = \mathrm{O}(x^{-n}n^{-1}) \qquad \textit{and} \qquad a(n) \sim cx^{-n}n^{-\mu},$$

then all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density.

1·3. *Conditions on the component counting function*

The first examples of additive number systems that satisfy the hypotheses of Compton's density theorem were provided by Knopfmacher et al. [**10**]. Their assumptions on the additive number system invoke the counting function $p(n)$ of the set of irreducible elements rather than $a(n)$. These asymptotics were generalized by Bell and Burris [**2**] as a consequence of the following general result.

THEOREM 1·4 (Bell and Burris [**2**]). *Let* $\mathbf{S}(x) := \sum_{n=0}^{\infty} s(n)x^n$ *be a power series with non-negative coefficients* $s(n)$ *and let* $\mathbf{S}^\star(x) := \sum_{n=0}^{\infty} s^\star(n)x^n$*, where* $s^\star(0) := 0$ *and* $s^\star(n) := \sum_{jk=n} s(j)/k$ *for* $n \geqslant 1$*, be its star transformation. Let* $\mathbf{T}(x) := \sum_{n=0}^{\infty} t(n)x^n$ *be the exponentiation of* $\mathbf{S}^\star(x)$*. If there is a* $0 < x < 1$ *with*

$$\lim_{n\to\infty} \frac{s(n-1)}{s(n)} = x \qquad \text{and} \qquad \liminf_{n\to\infty} ns(n)x^n > 1,$$

then it follows that $t(n-1)/t(n) < x$ *for all n large enough, and that*

$$\lim_{n\to\infty} \frac{t(n-1)}{t(n)} = x.$$

In the context of additive number systems, Bell and Burris then derive the theorem below, setting $s(n) := p(n)$ and $t(n) := a(n)$.

THEOREM 1·5 (Bell and Burris [**2**]). *Let* $\mathcal{A}$ *be an additive number system with the property that*

$$\lim_{n\to\infty} \frac{p(n-1)}{p(n)} = x \in (0,1) \qquad \text{and} \qquad \liminf_{n\to\infty} np(n)x^n > 1.$$

Then $\mathcal{A}$ *is reduced with radius of convergence x and* $a(n)$ *satisfies condition* (1·1).

Bell and Burris also show that if, for some $0 < x < 1$,

$$p(n) \sim cx^{-n}n^{-\mu}, \tag{1·2}$$

where either (i) $-\infty < \mu < 1$ and $c > 0$, or (ii) $\mu = 1$ and $c > 1$, then their conditions of Theorem 1·5 are satisfied. Granovsky and Stark [**8**] and Stark [**14**] have shown that some weakenings of (1·2), cases (i) and (ii), also lead to the conclusions of Theorem 1·5.

THEOREM 1·6 (Granovsky and Stark [**8**]). *Let* $\mathcal{A}$ *be an additive number system such that, for constants* $c_1, c_2, r > 0$, $0 < \varepsilon \leqslant r/3$ *and* $0 < x < 1$,

$$c_1 x^{-n} n^{(2r)/3+\varepsilon-1} \leqslant p(n) \leqslant c_2 x^{-n} n^{r-1} \qquad \text{for all } n \in \mathbb{N}.$$

Then $\mathcal{A}$ *is reduced with radius of convergence x, and* $a(n)$ *satisfies condition* (1·1).

THEOREM 1·7 (Stark [**14**]). *Let* $\mathcal{A}$ *be an additive number system such that*

$$1 < \inf_{n\in\mathbb{N}} np(n)x^n \leqslant \sup_{n\in\mathbb{N}} np(n)x^n < \infty,$$

where $0 < x < 1$*. Then* $\mathcal{A}$ *is reduced with radius of convergence x, and* $a(n)$ *satisfies condition* (1·1).

1·4. *Quasi-logarithmic additive number systems*

In this paper we consider a different weakening of condition (1·2), case (ii). We require that there are constants $0 < x < 1$ and $\theta > 0$, such that the sequence $np(n)x^n$, $n \in \mathbb{N}$,

 https://doi.org/10.1017/S0305004107000862

converges to θ "in an averaging sense", and such that "not too few" of the values $p(n)$ are strictly positive. These conditions are made precise in Subsection 2·2 as condition $\mathsf{C}_1(\theta)$ and C_2. They allow the counting function $p(n)$, for n large enough, to be, for example, of the form

$$p(n) \sim x^{-n}n^{-1}, \tag{1·3}$$

$$p(n) = \lfloor (1+\cos n)x^{-n}n^{-1} \rfloor, \tag{1·4}$$

$$p(n) = (1/2 + (\log n)^{-1})x^{-n}n^{-1} + \mathrm{O}(1), \tag{1·5}$$

$$p(n) = \lfloor \theta_n x^{-n}n^{-1} \rfloor, \tag{1·6}$$

where, in (1·6), θ_n converges on some set $I := \{a + kb, a + kb + 1 : k \in \mathbb{Z}_+\}$, with $a \in \mathbb{Z}_+$ and $b \in \mathbb{N}$, and where $\theta_n := 0$ for $n \in \mathbb{N} \setminus I$. Note that (1·3) is referred to as the *classical abstract prime number theorem* (cf. Knopfmacher and Zhang [**11**, section 3]).

If $\mathcal{A}$ is an additive number system where, for some $0 < x < 1$, $np(n)x^n$, $n \in \mathbb{N}$, satisfies conditions $\mathsf{C}_1(\theta)$ and C_2, the elements of $A(n) = \{\omega \in A : \|\omega\| = n\}$ are quasi-logarithmic multisets of total weight n, as introduced in Nietlispach [**13**]. We then refer to $\mathcal{A}$ as a (x, θ)-*quasi-logarithmic* additive number system. We show that, in a quasi-logarithmic additive number system, all partition sets have asymptotic density which equals the Dirichlet density (Theorem 3·1), and obtain a monadic second-order limit law for adequate classes $\mathcal{K}$ of finite relational structures whose associated additive number system $\mathcal{A}_{\mathcal{K}}$ is quasi-logarithmic (Corollary 3·4).

To prove Theorem 3·1, we first show in Section 2 that an (x, θ)-quasi-logarithmic additive number system satisfies

$$a(n) \sim cx^{-n}n^{\theta-1}\ell(n), \tag{1·7}$$

where $c > 0$ is a constant, and $\ell(n)$ is a specific slowly varying function (see (2·13) and (2·14)). By definition, a function $\ell\colon \mathbb{N} \to \mathbb{R}$ is *slowly varying (at infinity)* if

$$\lim_{n\to\infty} \frac{\ell(\lfloor \lambda n \rfloor)}{\ell(n)} = 1 \qquad \text{for all } \lambda > 0.$$

Such a function satisfies (cf. Bingham et al. [**3**, proposition 1·3·6 and lemma 1·9·6])

$$\lim_{n\to\infty} n^{\varepsilon}\ell(n) = \infty \quad \text{and} \quad \lim_{n\to\infty} n^{-\varepsilon}\ell(n) = 0 \qquad \text{for any } \varepsilon > 0, \tag{1·8}$$

and

$$\lim_{n\to\infty} \frac{\ell(n-1)}{\ell(n)} = 1. \tag{1·9}$$

The proof of (1·7) relies on probabilistic methods. We consider the vector of component counts

$$\left(C_1^{(n)}, \ldots, C_n^{(n)}\right) : A(n) \longrightarrow \mathbb{Z}_+^n,$$

where $A(n) = \{\omega \in A : \|\omega\| = n\}$ and where $C_i^{(n)}(\omega)$ is the number of components in $\omega \in A(n)$ of norm i, as a random vector on $A(n)$ endowed with the uniform probability measure. Then its distribution is compared with the distribution of an associated sequence of independent $\mathbb{Z}_+$-valued random variables $Z_1, \ldots, Z_n$, conditioned on $T_n := \sum_{i=1}^n iZ_i = n$. It is shown in Nietlispach [**13**] using coupling arguments that, under $\mathsf{C}_1(\theta)$ and C_2, the unconditioned distribution of T_n satisfies

$$\mathbb{P}[T_n = n] \sim \rho_\theta(1)n^{-1}, \tag{1·10}$$

where ρ_θ is the density function of the Dickman distribution with parameter $\theta > 0$ (cf. Arratia et al. [**1**, section 4·2] for an account of this distribution). Indeed, condition C_2 is required only for the coupling arguments. We then use (1·10) to derive (1·7). In Section 3 we show that, if $\theta > 1$, the local counting function $a(n)$, given as in (1·7), satisfies the requirements of Compton's density theorem. To prove Theorem 3·1 in the case $0 < \theta \leqslant 1$, we give an extension of Woods's density theorem to situations where $a(n)$ has the form (1·7), by extending the underlying Tauberian theorem of Woods [**15**].

Our density theorem for quasi-logarithmic additive number systems holds true in situations which are not covered by the results obtained so far:

(i) an (x, θ)-quasi-logarithmic additive number system with parameter $\theta < 1$ is not covered by Compton's density theorem;

(ii) if in addition the slowly varying function $\ell(n)$ does not converge to a positive constant, the form of the counting function $a(n)$ in (1·7) yields additive number systems to which Woods's density theorem does not apply either;

(iii) if we have an (x, θ)-quasi-logarithmic additive number system such that $\theta > 1$ and $\liminf_{n\to\infty} np(n)x^n \leqslant 1$, then the requirements of Compton's density theorem are true, but $p(n)$ does not satisfy the conditions of Theorems 1·5, 1·6 or 1·7.

2. *Asymptotics of the local counting function $a(n)$*

2·1. *A conditioning relation for component counts*

Let $\mathcal{A}$ be an additive number system. For each $n \in \mathbb{N}$ with $A(n) \neq \varnothing$ we endow the finite set $A(n)$ with the uniform distribution. Let $C_i^{(n)}(\omega)$ be the number of indecomposable elements of norm i in the unique decomposition of $\omega \in A(n)$. Clearly, $\sum_{i=1}^n iC_i^{(n)}(\omega) = n$, so that the random variables $C_1^{(n)}, \ldots, C_n^{(n)}$ are not independent. However, the joint distribution of these random variables can be expressed as a conditional joint distribution of independent random variables. To do so, let $0 < x < 1$ and let $\{Z_i(x)\}_{i\in\mathbb{N}}$ be a sequence of mutually independent random variables, where $Z_i(x)$ has the negative binomial $\mathrm{NB}(p(i), 1 - x^i)$-distribution for each i such that $p(i) > 0$,

$$\mathbb{P}[Z_i(x) = k] := \binom{p(i) + k - 1}{k} x^{ik}(1 - x^i)^{p(i)} \qquad \text{for } k \in \mathbb{Z}_+,$$

and where $Z_i(x) := 0$ for each i with $p(i) = 0$. Then we define

$$T_n(x) := \sum_{i=1}^n iZ_i(x) \qquad \text{for all } n \in \mathbb{N}, \tag{2·1}$$

and set $T_0(x) := 0$. We use the following conditioning relation for the vector of component counts of multisets $\omega \in A(n)$.

LEMMA 2·1 (Arratia et al. [**1**]). *Let $n \in \mathbb{N}$ with $a(n) > 0$ and let $0 < x < 1$. If $\mathbb{P}[T_n(x) = n] > 0$, we have*

$$\mathcal{L}\big(C_1^{(n)}, \ldots, C_n^{(n)}\big) = \mathcal{L}(Z_1(x), \ldots, Z_n(x) \mid T_n(x) = n). \tag{2·2}$$

To express the counting function $a(n)$ in terms of $p(1), \ldots, p(n)$, we introduce a new sequence $\{Z_i^*(x)\}_{i\in\mathbb{N}}$ of independent random variables as follows. Let $Z_i^*(x) := Z_i(x)$ if

$p(i) > 0$, and let $Z_i^*(x)$ be negative binomially NB$(1, 1 - x^i)$-distributed if $p(i) = 0$. We define

$$T_n^*(x) := T_{n-1}(x) + nZ_n^*(x) \qquad \text{for each } n \in \mathbb{N}. \tag{2·3}$$

LEMMA 2·2. *Let $n \in \mathbb{N}$ have $a(n) > 0$, and let $0 < x < 1$. If $\mathbb{P}[T_n(x) = n] > 0$, we have*

$$a(n) = \frac{\mathbb{P}[T_n^*(x) = n]}{x^n(1 - x^n)^{p(n)\vee 1}} \prod_{i=1}^{n-1}(1 - x^i)^{-p(i)} - \delta_n, \tag{2·4}$$

where $\delta_n := \mathbf{1}\{p(n) = 0\}$.

Proof. Fix $n \in \mathbb{N}$ with $a(n) > 0$, and $0 < x < 1$. Assume that $\mathbb{P}[T_n(x) = n] > 0$. We distinguish two cases, $p(n) > 0$ and $p(n) = 0$.

(i) First, assume that $p(n) > 0$. The conditioning relation (2·2) and the independence of the random variables $Z_1(x), \ldots, Z_n(x)$ yield

$$\begin{aligned}\frac{p(n)}{a(n)} &= \mathbb{P}[C_n^{(n)} = 1]\\ &= \mathbb{P}[(Z_1(x), \ldots, Z_{n-1}(x), Z_n(x)) = (0, \ldots, 0, 1)|T_n(x) = n]\\ &= \frac{\mathbb{P}[Z_n(x) = 1]}{\mathbb{P}[T_n(x) = n]} \prod_{i=1}^{n-1} \mathbb{P}[Z_i(x) = 0]\\ &= \frac{p(n)x^n(1 - x^n)^{p(n)}}{\mathbb{P}[T_n(x) = n]} \prod_{i=1}^{n-1}(1 - x^i)^{p(i)},\end{aligned}$$

and this implies equation (2·4), since $T_n^*(x) = T_n(x)$ and $\delta_n = 0$.

(ii) Now, assume that $p(n) = 0$. Here, $\mathbb{P}[T_n^*(x) = n] \geqslant \mathbb{P}[Z_n^*(x) = 1] > 0$, since $Z_n^*(x) \sim$ NB$(1, 1 - x^n)$. We define a new additive number system, $\mathcal{A}^* := (A^*, +^*, 0^*, P^*, \|\cdot\|^*)$ say, as the direct sum of $\mathcal{A}$ and an auxiliary additive number system that contains only one indecomposable element, which has norm n. Let $a^*(n)$ be the counting function of A^*, and $p^*(n)$ that of P^*. By construction we have $a^*(n) = a(n) + 1$, $p^*(i) = p(i)$ for $i \neq n$, and, $p^*(n) = p(n) + 1 = 1$. We endow $A^*(n)$ with the uniform probability measure, and consider random variables $C_i^{*(n)}$ that count the number of components of norm i in the sum decomposition of elements in $A^*(n)$. Lemma 2·1 yields

$$\mathcal{L}\left(C_1^{*(n)}, \ldots, C_n^{*(n)}\right) = \mathcal{L}(Z_1(x), \ldots, Z_{n-1}(x), Z_n^*(x)|T_n^*(x) = n),$$

where $Z_n^*(x) \sim$ NB$(1, 1 - x^n)$, since $p(n) = 0$. We argue much as in i), and conclude that

$$\begin{aligned}\frac{1}{a(n) + 1} = \frac{p^*(n)}{a^*(n)} &= \frac{p^*(n)x^n(1 - x^n)^{p^*(n)}}{\mathbb{P}[T_n^*(x) = n]} \prod_{i=1}^{n-1}(1 - x^i)^{p(i)}\\ &= \frac{x^n(1 - x^n)}{\mathbb{P}[T_n^*(x) = n]} \prod_{i=1}^{n-1}(1 - x^i)^{p(i)}.\end{aligned}$$

This yields equation (2·4), and finishes the proof.

2·2. *Conditions* $\mathsf{C}_1(\theta)$ *and* C_2

To examine the limiting behaviour of the local counting function $a(n)$ given as in (2·4), it is first necessary to establish the asymptotic behaviour of $\mathbb{P}[T_n^*(x) = n]$, as $n \to \infty$. We therefore formalize conditions $\mathsf{C}_1(\theta)$ and C_2 from Subsection 1·4 on the component counting function $p(i)$. We start with an arbitrary non-negative sequence $\{\theta_i\}_{i\in\mathbb{N}}$, instead of $\{ip(i)x^i\}_{i\in\mathbb{N}}$.

Definition 2·3. Given a constant $\theta \geqslant 0$ and a sequence $\{m_n\}_{n\in\mathbb{N}}$ of positive integers such that $m_n \to \infty$ and $m_n = \mathrm{o}(n)$, a sequence $\{\theta_i\}_{i\in\mathbb{N}}$ of non-negative real numbers satisfies $\mathsf{C}_1(\theta, \{m_n\}_{n\in\mathbb{N}})$ if

$$\theta_{\sup} := \sup_{i\in\mathbb{N}} \theta_i < \infty$$

and

$$\tilde{\theta}^{(n)}_{\max} := \max_{j\in J_n} \left| \frac{1}{m_n} \sum_{i=1}^{m_n} \theta_{jm_n+i} - \theta \right| \xrightarrow{n\to\infty} 0, \tag{2·5}$$

where $J_n := \{j \in \mathbb{Z}_+ : 0 \leqslant j \leqslant \lfloor n/m_n \rfloor\}$. If (2·5) holds true for every sequence $\{m_n\}_{n\in\mathbb{N}}$ as above, we say that $\{\theta_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{C}_1(\theta)$.

For details on the following examples of sequences that satisfy condition $\mathsf{C}_1(\theta)$ we refer to Nietlispach [**13**].

Example 2·4.

(i) Let $\{\theta_i\}_{i\in\mathbb{N}}$ converge to some $\eta > 0$ on the infinite set $I := \{i \in \mathbb{N} : \theta_i > 0\}$. Let $\{m_n\}_{n\in\mathbb{N}}$ be a sequence as in Definition 2·3. We define

$$l_n(j) := |\{1 \leqslant i \leqslant m_n : \theta_{jm_n+i} = 0\}| \qquad \text{for all } n \in \mathbb{N} \text{ and } j \in \mathbb{Z}_+,$$

and we assume that there is a $0 \leqslant x < 1$, such that

$$\max_{j\in J_n} \left| \frac{l_n(j)}{m_n} - x \right| \xrightarrow{n\to\infty} 0.$$

Then $\{\theta_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{C}_1(\theta, \{m_n\}_{n\in\mathbb{N}})$ with $\theta := (1-x)\eta$.

(ii) If $\{\theta_i\}_{i\in\mathbb{N}}$ is the integer skeleton of a non-negative function with rational period, or of a non-negative function with irrational period p and bounded variation on a closed interval of length p, then $\{\theta_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{C}_1(\theta)$ for some $\theta \geqslant 0$.

(iii) Starting with sequences as in (i) or (ii), more complicated examples can be constructed by noting that, if two sequences $\{\theta_i\}_{i\in\mathbb{N}}$ and $\{\theta'_i\}_{i\in\mathbb{N}}$ satisfy $\mathsf{C}_1(\theta, \{m_n\}_{n\in\mathbb{N}})$ and $\mathsf{C}_1(\theta', \{m_n\}_{n\in\mathbb{N}})$, respectively, then by the triangle inequality their sum satisfies condition $\mathsf{C}_1(\theta + \theta', \{m_n\}_{n\in\mathbb{N}})$. Moreover, if $\theta_i \geqslant \theta'_i$ for all $i \in \mathbb{N}$ and $\theta \geqslant \theta'$, then $\{\theta_i - \theta'_i\}_{i\in\mathbb{N}}$ satisfies $\mathsf{C}_1(\theta - \theta', \{m_n\}_{n\in\mathbb{N}})$.

The following lemma plays an important role in establishing properties of quasi-logarithmic additive number systems.

LEMMA 2·5. *Let* $\{\theta_i\}_{i\in\mathbb{N}}$ *satisfy condition* $\mathsf{C}_1(\theta, \{m_n\}_{n\in\mathbb{N}})$. *Then*

$$\ell(n) := \exp\left(\sum_{i=1}^{n} \frac{\theta_i - \theta}{i}\right), \qquad n \in \mathbb{N},$$

is slowly varying at infinity. Furthermore, if $m_n \leqslant n$, we have uniformly

$$\max\left\{\frac{\ell(n)}{\ell(l)}, \frac{\ell(l)}{\ell(n)}\right\} \leqslant (\chi_1 m_n)^{\theta_{\sup}} \left(\frac{n}{l \vee 1}\right)^{\tilde{\theta}^{(n)}_{\max}} \qquad \text{for } 0 \leqslant l < n, \tag{2·6}$$

where $\chi_1 := 2\mathrm{e}^{\zeta(2)+2}$, and

$$\max\left\{\frac{\ell(n)}{\ell(l)}, \frac{\ell(l)}{\ell(n)}\right\} \leqslant \chi_2^{\theta_{\sup}} \left(\frac{n}{l}\right)^{\tilde{\theta}^{(n)}_{\max}} \qquad \text{for } m_n \leqslant l < n, \tag{2·7}$$

where $\chi_2 := 4\mathrm{e}^{\zeta(2)+1}$; ζ is the Riemann Zeta function.

Proof. To show that ℓ is a slowly varying function, we use the notation $\tilde{\theta}^{(n,j)} := m_n^{-1} \sum_{i=1}^{m_n} \theta_{jm_n+i}$ for $n \in \mathbb{N}$ and $j \in \mathbb{Z}_+$. Let $0 < \lambda < 1$. Since $m_n = \mathrm{o}(n)$, we find non-negative integers a_n, b_n and $r_n, s_n < m_n$, such that $\lfloor \lambda n \rfloor = a_n m_n + r_n$ and $n = b_n m_n + s_n$. Note that $a_n \to \infty$, $b_n \to \infty$ and $a_n/b_n \to \lambda$. Now $1 \leqslant a_n < b_n$ for all n large enough. For these n it follows that

$$\begin{aligned}
\sum_{i=\lfloor \lambda n \rfloor+1}^{n} \frac{\theta_i - \theta}{i} &= \sum_{j=a_n}^{b_n-1} \sum_{i=1}^{m_n} \frac{\theta_{jm_n+i} - \theta}{jm_n + i} + \sum_{i=1}^{s_n} \frac{\theta_{b_n m_n+i} - \theta}{b_n m_n + i} - \sum_{i=1}^{r_n} \frac{\theta_{a_n m_n+i} - \theta}{a_n m_n + i} \\
&= \underbrace{\sum_{j=a_n}^{b_n-1} \sum_{i=1}^{m_n} \left(\theta_{jm_n+i} - \tilde{\theta}^{(n,j)}\right) \left(\frac{1}{jm_n + i} - \frac{1}{jm_n}\right)}_{U} + \sum_{j=a_n}^{b_n-1} \sum_{i=1}^{m_n} \frac{\tilde{\theta}^{(n,j)} - \theta}{jm_n + i} \\
&\quad + \sum_{i=1}^{s_n} \frac{\theta_{b_n m_n+i} - \theta}{b_n m_n + i} - \sum_{i=1}^{r_n} \frac{\theta_{a_n m_n+i} - \theta}{a_n m_n + i},
\end{aligned}$$

where the term U uses $\sum_{i=1}^{m_n} (\theta_{jm_n+i} - \tilde{\theta}^{(n,j)}) = 0$. Since $|\theta_{jm_n+i} - \tilde{\theta}^{(n,j)}| \leqslant \theta_{\sup}$, and

$$\sum_{i=k+1}^{l} \frac{1}{i} \leqslant \log(l/k) \qquad \text{for } 1 \leqslant k < l, \tag{2·8}$$

we conclude that

$$\begin{aligned}
\left| \sum_{i=\lfloor \lambda n \rfloor+1}^{n} \frac{\theta_i - \theta}{i} \right| &\leqslant \theta_{\sup} \sum_{j=a_n}^{b_n-1} \frac{1}{j^2} + \tilde{\theta}^{(n)}_{\max} \sum_{i=a_n m_n+1}^{b_n m_n} \frac{1}{i} + \frac{\theta_{\sup} s_n}{b_n m_n} + \frac{\theta_{\sup} r_n}{a_n m_n} \\
&\leqslant \theta_{\sup} \sum_{j=a_n}^{b_n-1} \frac{1}{j^2} + \tilde{\theta}^{(n)}_{\max} \log(b_n/a_n) + \frac{\theta_{\sup}}{b_n} + \frac{\theta_{\sup}}{a_n} \\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

This entails that

$$\lim_{n\to\infty} \frac{\ell(\lfloor \lambda n \rfloor)}{\ell(n)} = \lim_{n\to\infty} \exp\left(-\sum_{i=\lfloor \lambda n \rfloor+1}^{n} \frac{\theta_i - \theta}{i}\right) = 1$$

for all $0 < \lambda < 1$. It follows from Bingham et al. [**3**, theorem 1·4·1] that ℓ is slowly varying.

Inequalities (2·6) and (2·7) are proved with similar arguments. We write $l = am_n + r$ and $n = bm_n + s$, with non-negative integers a, b and $r, s < m_n$, and we apply (2·8) and $\sum_{i=1}^{n} 1/i \leqslant \log n + 1$ in the calculations below.

If $a < b$ we obtain

$$\left|\sum_{i=l+1}^{n} \frac{\theta_i - \theta}{i}\right| \leqslant \theta_{\sup} \sum_{j=a+1}^{b-1} \frac{1}{j^2} + \tilde{\theta}_{\max}^{(n)} \sum_{i=l+1}^{n} \frac{1}{i}$$

$$+ \theta_{\sup} \sum_{i=1}^{s} \frac{1}{bm_n + i} + \theta_{\sup} \sum_{i=1}^{m_n - r} \frac{1}{am_n + r + i}$$

$$\leqslant \theta_{\sup} \sum_{j=a+1}^{b-1} \frac{1}{j^2} + \tilde{\theta}_{\max}^{(n)} \sum_{i=l+1}^{n} \frac{1}{i} + \theta_{\sup} \sum_{i=m_n+1}^{2m_n} \frac{1}{i} + \theta_{\sup} \sum_{i=1}^{m_n} \frac{1}{i}$$

$$\leqslant \theta_{\sup}\zeta(2) + \tilde{\theta}_{\max}^{(n)}(\log(n/(l \vee 1)) + 1) + \theta_{\sup} \log 2 + \theta_{\sup}(\log m_n + 1).$$

If $a = b$ it follows that

$$\left|\sum_{i=l+1}^{n} \frac{\theta_i - \theta}{i}\right| \leqslant \theta_{\sup}(\log m_n + 1).$$

This yields (2·6).

If $l \geqslant m_n$, we have $a \geqslant 1$ and thus

$$\sum_{i=1}^{m_n - r} \frac{1}{am_n + r + i} \leqslant \sum_{i=m_n+1}^{2m_n} \frac{1}{i} \leqslant \log 2,$$

from which we conclude (2·7). This finishes the proof.

We turn to the second condition, C_2. As before, let $\{\theta_i\}_{i \in \mathbb{N}}$ be a sequence of non-negative integers. This condition generalizes the idea that there is an $\varepsilon > 0$, such that we find either

(i) two arithmetical progressions $\{jr\}_{j \in \mathbb{N}}$ and $\{js\}_{j \in \mathbb{N}}$, where $s < r$ are coprime, such that $\theta_{ju} > \varepsilon$ for all $j \in \mathbb{N}$ and $u \in \{r, s\}$,

(ii) or arithmetical progressions $\{jr\}_{j \in \mathbb{N}}$ and $\{jr - s\}_{j \in \mathbb{N}}$, where $s < r$ are coprime, such that $\theta_{jr} > \varepsilon$ and $\theta_{jr-s} > \varepsilon$ for all $j \in \mathbb{N}$.

We relax assumptions (i) and (ii) by allowing "gaps" to appear within the arithmetical progressions. We therefore consider the following auxiliary definition.

Definition 2·6. Let $u \in \mathbb{N}$, $N \in \mathbb{Z}_+$ and $\varepsilon > 0$. A set

$$B_j(N, u) := \{N + ju + 1, N + ju + 2, \ldots, N + (j+1)u\}, \qquad \text{where } j \in \mathbb{Z}_+,$$

is a (ε, N, u)-*good block*, if $\theta_{N+(j+1)u} > \varepsilon$. The set $B_j(N, u)$ is called a (ε, N, u, v)-*good block* if it is a (ε, N, u)-good block, and if there is a $v \in \mathbb{N}$, $v < u$, such that $\theta_{N+(j+1)u-v} > \varepsilon$.

Now we can state condition C_2 in its general form, which includes a third case, (iii), in addition to the general forms of (i) and (ii).

Definition 2·7. Let $\{\theta_i\}_{i \in \mathbb{N}}$ be a sequence of non-negative integers. The sequence satisfies condition C_2, if one of the following properties holds true:

(i) there is a constant $\varepsilon > 0$, coprime natural numbers $s < r$, and non-negative integers N_r, N_s, with the property that for each $u \in \{r, s\}$ there are runs of (ε, N_u, u)-good blocks $B_j(N_u, u)$ in $\mathbb{N}$, such that the length of each such run is larger than the length of the longest gap between two consecutive runs of (ε, N_u, u)-good blocks;

(ii) we find a constant $\varepsilon > 0$, coprime natural numbers $s < r$, and a non-negative integer N, with the property that there are runs of (ε, N, r, s)-good blocks $B_j(N, r)$ in $\mathbb{N}$, such that the length of each such run is larger than the length of the longest gap between two consecutive runs of (ε, N, r, s)-good blocks;

(iii) there exists some $\varepsilon > 0$, such that $\theta_i > \varepsilon$ for every odd $i \in \mathbb{N}$.

A quasi-logarithmic additive number system $\mathcal{A}$ is defined by imposing the conditions $\mathsf{C}_1(\theta)$ and C_2 on the component counting function $p(n)$ as follows.

Definition 2·8. An additive number system $\mathcal{A}$ is called (x, θ)-*quasi-logarithmic* for constants $0 < x < 1$ and $\theta > 0$ if the sequence $\{np(n)x^n\}_{n\in\mathbb{N}}$ satisfies conditions $\mathsf{C}_1(\theta)$ and C_2. The constants x and θ are unique if they exist.

Remark 2·9. Note that we require that $\theta > 0$ in contrast to the definition of condition $\mathsf{C}_1(\theta)$, where $\theta = 0$ is allowed.

The additive number systems defined by (1·3), (1·4), (1·5) and (1·6) in Subsection 1·4 are indeed quasi-logarithmic. We show this for (1·4). It follows in this case that we have for all $i \in \mathbb{N}$

$$ip(i)x^i = 1 + \cos i - i\varepsilon_i(x)x^i \geqslant 0 \qquad \text{for suitable } 0 \leqslant \varepsilon_i(x) < 1.$$

The sequence $\{1 + \cos i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{C}_1(1)$ as in Example 2·4 (ii). Condition C_2 is satisfied in the form of Definition 2·7 (ii). Indeed, we can choose $\varepsilon > 0$ small enough such that $1 + \cos i \leqslant \varepsilon$ for at most one i every two periods, so that Definition 2·7 (ii) holds true with $(\varepsilon, 0, 2, 1)$-good blocks $B_j(0, 2) = \{2j + 1, 2j + 2\}$. Since $0 < x < 1$, $\{i\varepsilon_i(x)x^i\}_{i\in\mathbb{N}}$ converges to 0. Thus, $\{ip(i)x^i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{C}_1(1)$ by Example 2·4 (iii), and also condition C_2.

Example 2·10 (Mapping patterns). Consider the set $\mathrm{Map}(n)$ of all mappings from the set $\{1, \ldots, n\}$ to itself. A mapping f corresponds to a labelled digraph with edges $(i, f(i))$, $1 \leqslant i \leqslant n$, where every vertex has outdegree 1. The connected components are directed cycles of rooted labelled trees. In a mapping pattern we consider the underlying topology of such graphs. That is, we consider equivalence classes of mappings f under the relation defined by

$$f \sim g :\Leftrightarrow \text{There is a permutation } \pi \in S_n \text{ such that } f \circ \pi = \pi \circ g.$$

This gives rise to an additive number system $\mathcal{A} := (A, +, 0, P, \|\cdot\|)$, where $A(n) := \mathrm{Map}(n)/_\sim$ for all $n \in \mathbb{N}$, $+$ is the disjoint union of graphs, 0 is the empty graph, P is the set of connected graphs, and $\|\cdot\|$ gives the number of vertices in a graph. Meir and Moon [**12**] showed that

$$p(n) = (2n)^{-1}x^{-n} + \mathrm{O}\left(n^{-3/2}x^{-n}\right) \qquad \text{where } x \approx 0.3383.$$

Example 2·11 (Multisets of aperiodic necklaces). Words of length n over a fixed alphabet of size q form equivalence classes under rotation, called necklaces. They correspond to 2-regular graphs with coloured beads. Consider the set $\mathrm{Neck}_q(n)$ of all necklaces of *aperiodic* words of size n over an alphabet of size q. This construction yields naturally an additive number system $\mathcal{A} := (A, +, 0, P, \|\cdot\|)$ by setting $P(n) := \mathrm{Neck}_q(n)$ for all $n \in \mathbb{N}$. We define $+$, 0 and $\|\cdot\|$ as in Example 2·10. It follows that (cf. Arratia et al. [**1**, example 2·12], Knopfmacher and Zhang [**11**, section 3·1·1])

$$p(n) = n^{-1}x^{-n} + \mathrm{O}\left(n^{-1}x^{-n/2}\right) \qquad \text{where } x = 1/q.$$

Example 2·12.

(i) Every additive arithmetical semigroup, a terminology used by Knopfmacher and Zhang [**11**] for additive number systems, that satisfies the "classical" abstract prime number theorem (1·3), that is

$$p(n) \sim x^{-n}n^{-1} \qquad \text{for some } 0 < x < 1, \tag{2·9}$$

is a $(x, 1)$-quasi-logarithmic additive number system. Knopfmacher and Zhang [**11**, sections 1·1 and 3·1] give an exhaustive list of explicit additive arithmetical semigroups that satisfy (2·9). All of these examples satisfy (2·9) in the form

$$p(n) = x^{-n}n^{-1} + \mathrm{O}\big(x^{-n/2}n^{-1}\big).$$

(ii) An additive arithmetical semigroup where the "non-classical" abstract prime number theorem first examined by Indlekofer et al. [**9**] (cf. also Knopfmacher and Zhang [**11**, section 5]),

$$np(n)x^n = 1 + (-1)^{n+1} + \mathrm{o}(1) \qquad \text{for some } 0 < x < 1,$$

holds true, is a $(x, 1/2)$-quasi-logarithmic additive number system.

(iii) What is more, Knopfmacher and Zhang [**11**, examples 3·8·1 and 3·8·6] consider two analytical examples of additive arithmetic semigroups. Both are examples of $(x, 1)$-quasi-logarithmic additive number systems; the first satisfies

$$p(n) = \begin{cases} 2x^{-n}n^{-1} + \mathrm{O}(1) & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases} \qquad \text{for some } 0 < x < 1,$$

the second

$$p(n) = x^{-n}n^{-1} + \mathrm{O}(1) \qquad \text{for some } 0 < x < 1.$$

Remark 2·13. Additional quasi-logarithmic additive number systems can be constructed, by "perturbing" the additive number systems considered in Examples 2·10, 2·11 and 2·12. Components can be "deleted", or additional instances of a component can be introduced (e. g. by colouring).

2·3. *Asymptotic behaviour of $a(n)$ under $\mathsf{C}_1(\theta)$ and C_2*

Recall the definition of the random variables $T_n(x)$ in (2·1) and $T_n^*(x)$ in (2·3).

LEMMA 2·14. *Let $\mathcal{A}$ be an (x, θ)-quasi-logarithmic additive number system. We have*

$$\mathbb{P}[T_n^*(x) = n] \sim \mathbb{P}[T_n(x) = n] \sim \rho_\theta(1)n^{-1}, \tag{2·10}$$

where ρ_θ is the density of the Dickman distribution with parameter θ.

Proof. Let $\theta_i := ip(i)x^i$, $i \in \mathbb{N}$. Under our assumption on $\mathcal{A}$, $\{\theta_i\}_{i\in\mathbb{N}}$ satisfies conditions $\mathsf{C}_1(\theta)$ and C_2. Since $0 < x < 1$ and

$$i\mathbb{E}Z_i(x) = \frac{\theta_i}{(1 - x^i)} \geqslant \theta_i \qquad \text{for } i \in \mathbb{N},$$

the sequence $\{i\mathbb{E}Z_i(x)\}_{i\in\mathbb{N}}$ satisfies these conditions as well. Under condition C_2 on $\{i\mathbb{E}Z_i(x)\}_{i\in\mathbb{N}}$ we can invoke the local approximation theorem for the Dickman distribution

P_θ in Nietlispach [**13**] to conclude that

$$\lim_{n\to\infty} n\mathbb{P}[T_n(x) = k_n] = \rho_\theta(1), \tag{2·11}$$

for every integer sequence $k_n \sim n$. Note that we have $T_n(x) = T_n^*(x)$, if $p(n) > 0$, and

$$\mathbb{P}[T_n^*(x) = n] = \mathbb{P}[T_{n-1}(x) = n](1 - x^n) + \mathbb{P}[T_{n-1}(x) = 0]x^n(1 - x^n),$$

if $p(n) = 0$. Now (2·10) follows from (2·11), since $0 < x < 1$, and so $nx^n \to 0$.

THEOREM 2·15. *Let $\mathcal{A}$ be an (x, θ)-quasi-logarithmic additive number system. Then $\mathcal{A}$ is reduced. We have*

$$a(n) \sim cx^{-n}n^{\theta-1}\ell(n), \tag{2·12}$$

where

$$0 < c := \rho_\theta(1)e^{\theta\gamma}\left(\lim_{n\to\infty}\prod_{i=1}^{n}\left(e^{x^i}(1 - x^i)\right)^{-p(i)}\right) < \infty, \tag{2·13}$$

γ being Euler's constant and ρ_θ the density of the Dickman distribution, and

$$\ell(n) := \exp\left(\sum_{i=1}^{n}\frac{ip(i)x^i - \theta}{i}\right), \qquad n \in \mathbb{N} \tag{2·14}$$

is slowly varying at infinity. In particular, x is the radius of convergence of $\mathcal{A}$.

Proof. Let $\theta_i := ip(i)x^i$ for $i \in \mathbb{N}$. By assumption, $\{\theta_i\}_{i\in\mathbb{N}}$ satisfies condition C_2, which implies that $\mathcal{A}$ is reduced. This is clear under condition C_2 (iii). Under condition C_2 (i) and (ii) there are natural numbers j, k, l, m (not necessarily pairwise distinct) with $\theta_j, \theta_k, \theta_l, \theta_m > \varepsilon$, or equivalently $p(j), p(k), p(l), p(m) > 0$, such that $j - k = r$ and $l - m = s$, where $\gcd(r, s) = 1$. Now, if we had $g := \gcd(j, k, l, m) > 1$, then $g > 1$ would be a divisor of both r and s, which contradicts the assumption that r and s are coprime. Hence, $\gcd\{n \in \mathbb{N} : p(n) > 0\} = \gcd(j, k, l, m) = 1$, so that $\mathcal{A}$ is reduced.

Since $\mathcal{A}$ is reduced, $a(n) > 0$ for all $n \in \mathbb{N}$ large enough. Moreover, (2·10) entails that $\mathbb{P}[T_n(x) = n] > 0$ for all $n \in \mathbb{N}$ large enough. Therefore, we can invoke Lemma 2·2 and write

$$a(n) + \delta_n = \frac{\mathbb{P}[T_n^*(x) = n]}{x^n(1 - x^n)^{p(n)\vee 1 - p(n)}}\prod_{i=1}^{n}(1 - x^i)^{-p(i)}$$

for every $n \in \mathbb{N}$ large enough. Since $0 < x < 1$, we have

$$\lim_{n\to\infty}(1 - x^n)^{p(n)\vee 1 - p(n)} = 1.$$

Since also $\theta_{\sup} < \infty$, the product $\prod_{i=1}^{n}\left(e^{x^i}(1 - x^i)\right)^{-p(i)}$ converges to some constant $0 < c' < \infty$, as $n \to \infty$, and it follows that

$$\begin{aligned}\prod_{i=1}^{n}(1 - x^i)^{-p(i)} &\sim c'\exp\left(\sum_{i=1}^{n}p(i)x^i\right)\\ &= c'\exp\left(\sum_{i=1}^{n}\frac{\theta}{i}\right)\exp\left(\sum_{i=1}^{n}\frac{ip(i)x^i - \theta}{i}\right)\\ &\sim c'e^{\theta\gamma}n^\theta\ell(n).\end{aligned}$$

Then we obtain, invoking (2·10), that

$$a(n) + \delta_n \sim c x^{-n} n^{\theta-1} \ell(n),$$

where $c := c' e^{\theta\gamma} \rho_\theta(1) > 0$. Finally, (2·12) follows from

$$\lim_{n\to\infty} \frac{\delta_n}{x^{-n} n^{\theta-1} \ell(n)} = 0.$$

The slow variation of $\ell(n)$ is immediate from Lemma 2·5 and, since $\ell(n)$ is slowly varying, we conlude from (2·12) and (1·9) that

$$\lim_{n\to\infty} \frac{a(n-1)}{a(n)} = x,$$

so that x is the radius of convergence of $\mathcal{A}$.

3. *Asymptotic density in quasi-logarithmic additive number systems*

3·1. *A density theorem and a logical limit law*

Now we can state and prove our main result, a density theorem for quasi-logarithmic additive number systems.

THEOREM 3·1. *Let $\mathcal{A}$ be an (x, θ)-quasi-logarithmic additive number system. Then all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density.*

Proof. We treat the cases $0 < \theta \leqslant 1$ and $\theta > 1$ separately. For the first case we use Proposition 3·8 below, an extension of Woods's density theorem, whereas for the second case we invoke Compton's density theorem.

First, assume that $0 < \theta \leqslant 1$. Here, from Theorem 2·15, the requirements of Proposition 3·8 hold true with $\rho := x$, $\mu := 1 - \theta$ and $\theta_i := ip(i)x^i$, $i \in \mathbb{N}$. Hence, all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density.

Now, let $\theta > 1$. Theorem 2·15 implies that $0 < x < 1$ is the radius of convergence of $\mathcal{A}$. We show that there are constants $C, K > 0$, such that (1·1) holds true with $\rho := x$. Then Theorem 1·1 implies that all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density.

Let $k = n$. It follows from (2·12) and because $a(0) = 1$, that

$$\frac{a(0)}{a(n)} \sim \frac{1}{cn^{\theta-1}\ell(n)} x^n.$$

Since $\ell(n)$ is slowly varying by Theorem 2·15, and $\theta > 1$, (1·8) yields $n^{\theta-1}\ell(n) \to \infty$. Since $\mathcal{A}$ is reduced, by Theorem 2·15, there is a $K_0 > 0$ such that $a(n) \geqslant 1$ for all $n \geqslant K_0$. It follows that, for some $c_1 > 0$,

$$\frac{a(0)}{a(n)} \leqslant c_1 x^n \qquad \text{for all } n \geqslant K_0.$$

If $0 \leqslant k < n$, (2·12) implies that, for some $c_2 > 0$,

$$\frac{a(n-k)}{a(n)} \leqslant c_2 \left(\frac{n-k}{n}\right)^{\theta-1} \frac{\ell(n-k)}{\ell(n)} x^k \qquad \text{for all } n \geqslant K_0. \tag{3·1}$$

Since, by assumption, $\{\theta_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{C}_1(\theta, \{m_n\}_{n\in\mathbb{N}})$ for every sequence $\{m_n\}_{n\in\mathbb{N}}$

of positive integers with $m_n \to \infty$ and $m_n = \mathrm{o}(n)$, we choose $\{m_n\}_{n\in\mathbb{N}}$ such that there is a $K_1 > K_0$ with

$$m_n \leqslant n^{(\theta-1)/(\theta-1+2\theta_{\sup})} \qquad \text{for all } n \geqslant K_1. \tag{3·2}$$

Then we choose $K > K_1$ such that

$$\theta - 1 - \tilde{\theta}_{\max}^{(n)} \geqslant \frac{\theta-1}{2} \qquad \text{for all } n \geqslant K. \tag{3·3}$$

Let $n \geqslant K$. If $n - m_n < k < n$, we conclude from (3·1), using (2·6), (3·3) and (3·2), that

$$\begin{aligned} \frac{a(n-k)}{a(n)} &\leqslant c_2 \chi_1^{\theta_{\sup}} m_n^{\theta_{\sup}} \left(\frac{n-k}{n}\right)^{(\theta-1)/2} x^k \\ &\leqslant c_2 \chi_1^{\theta_{\sup}} m_n^{\theta_{\sup}} \left(\frac{m_n}{n}\right)^{(\theta-1)/2} x^k \\ &\leqslant c_2 \chi_1^{\theta_{\sup}} x^k. \end{aligned}$$

If $0 \leqslant k \leqslant n - m_n$, we obtain from (3·1), using (2·7) and (3·3), that

$$\frac{a(n-k)}{a(n)} \leqslant c_2 \chi_2^{\theta_{\sup}} \left(\frac{n-k}{n}\right)^{(\theta-1)/2} x^k \leqslant c_2 \chi_2^{\theta_{\sup}} x^k.$$

Now (1·1) follows with K as above, $C := \max\{c_1, c_2\chi_1^{\theta_{\sup}}, c_2\chi_2^{\theta_{\sup}}\}$ and $\rho := x$. This proves the theorem.

Remark 3·2. Compton's density theorem cannot be applied if $\theta < 1$. Indeed, it follows from (2·12) that there is a constant $c' > 0$, such that

$$\frac{a(1)}{a(n)} \geqslant c' \frac{1}{n^{\theta-1}\ell(n)} x^{n-1} \qquad \text{for all } n \in \mathbb{N},$$

and $n^{\theta-1}\ell(n) \to 0$ by (1·8), since $\theta < 1$. Thus, assumption (1·1) of Theorem 1·1 is violated.

An immediate consequence of Theorem 3·1 and Example 2·12 is the following result on additive number systems that satisfy an abstract prime number theorem.

COROLLARY 3·3. *Let $\mathcal{A}$ be an additive number system which satisfies an abstract prime number theorem in the form*

$$p(n) \sim x^{-n} n^{-1} \qquad \text{or} \qquad np(n)x^n = 1 + (-1)^{n+1} + \mathrm{o}(1),$$

for some $0 < x < 1$. Then all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density.

As a consequence of Theorem 3·1 and Theorem 1·2 we obtain the following logical limit law.

COROLLARY 3·4. *Let L be a finite, purely relational language, and let $\mathcal{K}$ be an adequate class of finite L-structures, such that the additive number system $\mathcal{A}_{\mathcal{K}}$ is quasi-logarithmic. Then $\mathcal{K}$ has a monadic second-order limit law.*

The additive number systems from Examples 2·10 and 2·11, and also the explicit examples mentioned in Example 2·12 (i) and (iii) are covered by Woods's density theorem already. In these cases, the slowly varying function $\ell(n)$ defined in (2·14), with $x := 1/q$ and $\theta := 1$, converges to a positive constant.

For an additive number system whose counting function $p(n)$ satisfies (1·6) neither Compton's nor Woods's density theorem holds true. Nonetheless, all partition sets have asymptotic density, as can be seen in the following example.

Example 3·5. Let $\mathcal{A}$ be an additive number system, where, for some $0 < x < 1$, $p(n)$ satisfies (1·5), that is

$$p(n) = (1/2 + (\log n)^{-1})x^{-n}n^{-1} + \mathrm{O}(1) \qquad \text{for all } n \in \mathbb{N}, n \geqslant 2.$$

We have

$$\theta_n(x) := np(n)x^n \xrightarrow{n\to\infty} 1/2,$$

so that $\mathcal{A}$ is (x, θ)-quasi-logarithmic with $\theta := 1/2$. By Theorem 3·1, all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density. It follows from Remark 3·2 that $\mathcal{A}$ does not satisfy the assumptions of Theorem 1·1. What is more we have, for some $K > 0$,

$$\left|\sum_{i=1}^{n} \frac{\theta_i(x) - \theta}{i} - \log\log n\right| \leqslant \left|\sum_{i=1}^{n} \frac{\theta_i(x) - \theta}{i} - \sum_{i=2}^{n} \frac{1}{i \log i}\right| + \left|\sum_{i=2}^{n} \frac{1}{i \log i} - \log\log n\right| \leqslant K.$$

This yields

$$\ell(n) := \exp\left(\sum_{i=1}^{n} \frac{\theta_i(x) - \theta}{i}\right) \asymp \log n,$$

so that (2·12) entails with $\theta = 1/2$ that

$$a(n) \asymp x^{-n}n^{-1/2}\ell(n) \asymp x^{-n}n^{-1/2}\log n.$$

Thus, $\mathcal{A}$ does not satisfy the requirements of Theorem 1·3 either.

3·2. *Extending Woods's results*

We prove an extension of a Tauberian theorem of Woods [**15**] and obtain a extension of Theorem 1·3 in this way.

LEMMA 3·6 (Extension of Woods's Tauberian theorem). *Let* $\mathbf{S}(x) := \sum_{n=0}^{\infty} s(n)x^n$, $\mathbf{T}(x) := \sum_{n=0}^{\infty} t(n)x^n$ *be two power series, and let* $\mathbf{R}(x) := \sum_{n=0}^{\infty} r(n)x^n$ *be the Cauchy product of* $\mathbf{S}(x)$ *and* $\mathbf{T}(x)$*, that is*

$$r(n) := \sum_{j+k=n} s(j)t(k) \qquad \text{for all } n \in \mathbb{Z}_+.$$

If $\mathbf{S}(\rho)$ *converges absolutely at* ρ *for some* $0 < \rho \leqslant 1$, *and if for some* $c > 0$ *and* $0 \leqslant \mu < 1$

$$s(n) = \mathrm{O}(\rho^{-n}n^{-1}), \tag{3·4}$$
$$t(n) \sim c\rho^{-n}n^{-\mu}\ell(n), \tag{3·5}$$

where

$$\ell(n) := \exp\left(\sum_{i=1}^{n} \frac{\theta_i - \theta}{i}\right), \qquad n \in \mathbb{N},$$

and $\{\theta_i\}_{i\in\mathbb{N}}$ *is a non-negative sequence that satisfies condition* $\mathsf{C}_1(\theta)$*, then*

$$\lim_{n\to\infty} \frac{r(n)}{t(n)} = \mathbf{S}(\rho) = \lim_{x\to\rho^-} \frac{\mathbf{R}(x)}{\mathbf{T}(x)}. \tag{3·6}$$

Remark 3·7. In contrast to our assumptions, Woods [**15**] requires that $t(n) \sim c\rho^{-n} n^{-\mu}$ for some constants $c > 0$ and $-\infty < \mu < 1$.

Proof of Lemma 3·6. We mimic the proof of Woods's Tauberian theorem, as found in Burris [**4**, appendix E].

The second equality in (3·6) is immediate from the fact that $\mathbf{S}(\rho)$ converges absolutely, and that, from (3·5) and the slow variation of $\ell(n)$, ρ is the radius of convergence of $\mathbf{T}(x)$. In this case $\mathbf{R}(x)$ is equal the usual product $\mathbf{S}(x) \cdot \mathbf{T}(x)$ for all $0 \leqslant x < \rho$, and $\mathbf{S}(x)$ is continuous on $[0, \rho]$.

To prove the first equality of (3·6) we consider a change of variable $\rho x \mapsto x$. Then we have $\rho = 1$, and we have to show that

$$\lim_{n\to\infty} \left| \mathbf{S}(1) - \frac{r(n)}{t(n)} \right| = 0.$$

We introduce

$$R_n := \sum_{k>\sqrt{n}} |s(k)| \qquad \text{and} \qquad M_n := n\sqrt{R_n} + \sqrt{n}.$$

Note that $R_n \to 0$, since $\mathbf{S}(1)$ converges absolutely, and that $M_n \to \infty$ and $n - M_n > \sqrt{n}$ for all $n \in \mathbb{N}$ large enough. We also have

$$M_n = \mathrm{o}(n) \qquad \text{and} \qquad nR_n = \mathrm{o}(M_n).$$

For all n large enough for $n - M_n > \sqrt{n}$ to hold we consider

$$\begin{aligned} \left| \mathbf{S}(1) - \frac{r(n)}{t(n)} \right| &\leqslant \left| \mathbf{S}(1) - \sum_{0\leqslant k\leqslant n} s(k) \right| + \left| \sum_{0\leqslant k\leqslant n} s(k) - \frac{r(n)}{t(n)} \right| \\ &\leqslant \underbrace{\left| \mathbf{S}(1) - \sum_{0\leqslant k\leqslant n} s(k) \right|}_{U_1(n)} + \underbrace{\sum_{0\leqslant k\leqslant\sqrt{n}} \left| s(k)\left(1 - \frac{t(n-k)}{t(n)}\right) \right|}_{U_2(n)} + \underbrace{\sum_{\sqrt{n}<k\leqslant n} |s(k)|}_{U_3(n)} \\ &\quad + \underbrace{\sum_{\sqrt{n}<k\leqslant n-M_n} |s(k)| \frac{t(n-k)}{t(n)}}_{U_4(n)} + \underbrace{\sum_{n-M_n<k\leqslant n} |s(k)| \frac{t(n-k)}{t(n)}}_{U_5(n)}. \end{aligned}$$

The first and third expression, $U_1(n)$ and $U_3(n)$, converge to zero, since $\mathbf{S}(1)$ is absolutely convergent by assumption.

If f is a function on $\mathbb{N}$ and $A \subset \mathbb{N}$, we use $\arg\max_{k\in A} f(k)$ to denote the value $k \in A$ for which f is maximal. To bound $U_2(n)$, let

$$k_n := \arg \max_{0\leqslant k\leqslant\sqrt{n}} \left| 1 - \frac{t(n-k)}{t(n)} \right| = \mathrm{o}(n).$$

Note that $\ell(n)$ is slowly varying at infinity, by Lemma 2·5. Then we obtain from (3·5), with $\rho = 1$, that

$$\frac{t(n-k_n)}{t(n)} \sim \left(\frac{n}{n-k_n} \right)^{\mu} \frac{\ell(n-k_n)}{\ell(n)} \xrightarrow{n\to\infty} 1$$

Invoking the absolute convergence of $\mathbf{S}(1)$ once more, we conclude that

$$U_2(n) \leqslant \left(\sum_{k\geqslant 0}|s(k)|\right)\left|1-\frac{t(n-k_n)}{t(n)}\right| \xrightarrow{n\to\infty} 0.$$

To bound $U_4(n)$ from above, a similar argument is used. We introduce

$$k_n' := \arg\max_{\sqrt{n}<k\leqslant n-M_n} \frac{t(n-k)}{t(n)}$$

and conclude that

$$U_4(n) \leqslant R_n \frac{t(n-k_n')}{t(n)}. \tag{3·7}$$

Since $n-k_n' \geqslant M_n \to \infty$, we obtain from (3·5) with $\rho = 1$ that

$$\frac{t(n-k_n')}{t(n)} \sim \left(\frac{n}{n-k_n'}\right)^{\mu} \frac{\ell(n-k_n')}{\ell(n)}. \tag{3·8}$$

From the representation theorem for slowly varying sequences (cf. Bingham et al. [**3**, theorem 1·9·7]) it follows that

$$\frac{\ell(n-k_n')}{\ell(n)} \sim \exp\left(-\sum_{i=n-k_n'+1}^{n}\frac{\delta(i)}{i}\right) \leqslant \left(\frac{n}{n-k_n'}\right)^{\varepsilon(n)}, \tag{3·9}$$

where $\delta(i)\to 0$ and $\varepsilon(n) := \sup_{i\geqslant M_n}|\delta(i)| \to 0$. Using inequalities (3·8) and (3·9) together with (3·7) we conclude that, for some constant $c_1 > 0$,

$$U_4(n) \leqslant c_1 R_n \left(\frac{n}{n-k_n'}\right)^{\mu+\varepsilon(n)} \leqslant c_1 R_n \left(\frac{n}{M_n}\right)^{\mu+\varepsilon(n)}.$$

By assumption we have $0 \leqslant \mu < 1$. Since $\varepsilon(n) \to 0$, we have $\mu \leqslant \mu+\varepsilon(n) < 1$ for all $n \in \mathbb{N}$ large enough. For these n we have

$$U_4(n) \leqslant c_1 R_n^{1-\mu-\varepsilon(n)} \left(\frac{nR_n}{M_n}\right)^{\mu+\varepsilon(n)}. \tag{3·10}$$

Recall that $R_n \to 0$ and $nR_n = \mathrm{o}(M_n)$. Therefore, the right-hand side of (3·10), and thus $U_4(n)$, converges to zero for every $0 \leqslant \mu < 1$.

To bound $U_5(n)$, we rewrite this expression as

$$U_5(n) = \sum_{0\leqslant k<M_n} |s(n-k)|\frac{t(k)}{t(n)} \leqslant |s(n-k_n'')|\frac{t(0)}{t(n)} + |s(n-k_n'')|\sum_{1\leqslant k<M_n}\frac{t(k)}{t(n)}, \tag{3·11}$$

where

$$k_n'' := \arg\max_{0\leqslant k<M_n}|s(n-k)| = \mathrm{o}(n).$$

Under assumption (3·4), with $\rho = 1$, we have $|s(n-k_n'')| \sim 1/n$, so that, since $\ell(n)$ is slowly varying and therefore $n^{1-\mu}\ell(n) \to \infty$ for all $0 \leqslant \mu < 1$,

$$|s(n-k_n'')|\frac{t(0)}{t(n)} \sim \frac{t(0)}{n^{1-\mu}\ell(n)} \xrightarrow{n\to\infty} 0. \tag{3·12}$$

Furthermore, it follows from (3·5) that, for some constant $c_2 > 0$,

$$|s(n-k_n'')| \sum_{1\leqslant k<M_n} \frac{t(k)}{t(n)} \leqslant c_2 \frac{1}{n} \sum_{1\leqslant k<M_n} \left(\frac{n}{k}\right)^{\mu} \frac{\ell(k)}{\ell(n)}. \tag{3·13}$$

By assumption $\{\theta_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{C}_1(\theta, \{m_n\}_{n\in\mathbb{N}})$ for every sequence $\{m_n\}_{n\in\mathbb{N}}$ of positive integers with $m_n \to \infty$ and $m_n = \mathrm{o}(n)$. Here, we choose a sequence $\{m_n\}_{n\in\mathbb{N}}$ defined by

$$m_n := \left\lfloor \left(\frac{n}{M_n}\right)^{(1-\mu)/(2(\theta_{\sup}\vee 1))} \right\rfloor \vee 1 \qquad \text{for all } n \in \mathbb{N}.$$

Since $M_n = \mathrm{o}(n)$ and $0 < (1-\mu)/(2(\theta_{\sup}\vee 1)) < 1$ we have $m_n \to \infty$ and $m_n = \mathrm{o}(n)$ as required. Now invoke (2·6) with this choice of $\{m_n\}_{n\in\mathbb{N}}$. We obtain for all $n \in \mathbb{N}$, such that $M_n \leqslant n$,

$$\frac{\ell(k)}{\ell(n)} \leqslant \chi_1^{\theta_{\sup}} \left(\frac{n}{M_n}\right)^{(1-\mu)/2} \left(\frac{n}{k}\right)^{\tilde{\theta}_{\max}^{(n)}} \qquad \text{for all } 1 \leqslant k < M_n.$$

Combining this result with (3·13) implies that

$$|s(n-k_n'')| \sum_{1\leqslant k<M_n} \frac{t(k)}{t(n)} \leqslant c_2 \chi_1^{\theta_{\sup}} \frac{1}{n^{1-\mu-\tilde{\theta}_{\max}^{(n)}}} \left(\frac{n}{M_n}\right)^{(1-\mu)/2} \sum_{1\leqslant k<M_n} \frac{1}{k^{\mu+\tilde{\theta}_{\max}^{(n)}}}.$$

Since $0 \leqslant \mu < 1$ and $\tilde{\theta}_{\max}^{(n)} \to 0$ we have $\mu \leqslant \mu + \tilde{\theta}_{\max}^{(n)} < 1$ for all $n \in \mathbb{N}$ large enough, and for these n

$$\sum_{1\leqslant k<M_n} \frac{1}{k^{\mu+\tilde{\theta}_{\max}^{(n)}}} \leqslant \frac{M_n^{1-\mu-\tilde{\theta}_{\max}^{(n)}}}{1-\mu-\tilde{\theta}_{\max}^{(n)}}$$

holds true. Hence, we conclude that

$$|s(n-k_n'')| \sum_{1\leqslant k<M_n} \frac{t(k)}{t(n)} \leqslant \frac{c_2 \chi_1^{\theta_{\sup}}}{1-\mu-\tilde{\theta}_{\max}^{(n)}} \left(\frac{M_n}{n}\right)^{(1-\mu)/2-\tilde{\theta}_{\max}^{(n)}}. \tag{3·14}$$

The right-hand side of (3·14) converges to zero since $M_n = \mathrm{o}(n)$ and $\tilde{\theta}_{\max}^{(n)} \to 0$. Finally, combining (3·11) with (3·12) and (3·14) yields $U_5(n) \to 0$, which proves the proposition.

PROPOSITION 3·8 (Extension of Woods's density theorem). *Assume that $\mathcal{A}$ is an additive number system such that, for some $0 < \rho \leqslant 1$, $c > 0$ and $0 \leqslant \mu < 1$,*

$$p(n) = \mathrm{O}(\rho^{-n} n^{-1}), \tag{3·15}$$

$$a(n) \sim c\rho^{-n} n^{-\mu} \ell(n), \tag{3·16}$$

where

$$\ell(n) := \exp\left(\sum_{i=1}^{n} \frac{\theta_i - \theta}{i}\right), \qquad n \in \mathbb{N},$$

and $\{\theta_i\}_{i\in\mathbb{N}}$ satisfies condition $\mathsf{C}_1(\theta)$. Then all partition sets of $\mathcal{A}$ have asymptotic density which equals the Dirichlet density.

Proof. It follows from $a(n) \sim c\rho^{-n} n^{-\mu} \ell(n)$ and the slow variation of $\ell(n)$ that ρ is the radius of convergence of $\mathcal{A}$. The remainder of the proof is exactly the same as the proof of Woods's density theorem in Burris [**4**, appendix E], so we only give a short outline. Recall the definition of the asymptotic density δ and Dirichlet density ∂ in Subsection 1·2.

If $\rho = 1$ then Burris [**4**, theorem 4·2] states that $a(n-1)/a(n) \to 1$, as $n \to \infty$, is sufficient for all partition sets to have asymptotic density, which agrees with the Dirichlet density on these sets.

Now let $0 < \rho < 1$, and let B be a partition set. Note that the Dirichlet density $\partial(b)$ exists by Burris [**4**, theorem 3·40]. If $\partial(B) = 0$, then the asymptotic density $\delta(B)$ is zero as well (cf. Burris [**4**, corollary 5·7]).

It remains to prove the theorem in the case where $\partial(B) > 0$. Here, Lemma 3·6 is applied with special forms of the power series $\mathbf{S}(x) = \sum_{n=0}^{\infty} s(n)x^n$ and $\mathbf{T}(x) = \sum_{n=0}^{\infty} t(n)x^n$, namely

$$\mathbf{S}(x) := \bar{\mathbf{B}} * [1/\mathbf{A}](x) \qquad \text{and} \qquad \mathbf{T}(x) := \mathbf{A}(x),$$

where $\mathbf{A}(x)$ is the generating series of the additive number system $\mathcal{A}$, $[1/\mathbf{A}](x)$ is the power series expansion of $1/\mathbf{A}(x)$, $\bar{\mathbf{B}}(x) := \sum_{n=0}^{\infty} \bar{b}(n)x^n$ is the generating series of a special partition set $\bar{B}$, constructed from B (cf. Burris [**4**, p. 92]), and $*$ denotes the Cauchy product. In this special case, the conclusion (3·6) of Lemma 3·6 translates into

$$\lim_{n\to\infty} \frac{\bar{b}(n)}{a(n)} = \mathbf{S}(\rho) = \lim_{x\to\rho} \frac{\bar{\mathbf{B}}(x)}{\mathbf{A}(x)},$$

which means that $\delta(\bar{B}) = \partial(\bar{B})$. But $\delta(\bar{B}) = \delta(B)$ and $\partial(\bar{B}) = \partial(B)$, by Burris [**4**, lemma 5·12].

The assumptions of Lemma 3·6 hold true with our choices of $\mathbf{S}(x)$ and $\mathbf{T}(x)$. Indeed, since $t(n) = a(n)$ for all $n \in \mathbb{N}$, (3·16) immediately yields (3·5), and Burris [**4**, p. 278–279] proves that (3·15) implies (3·4). What is more, it follows from Burris [**4**, corollary 5·11 and p. 10 above] that $\mathbf{S}(\rho)$ converges absolutely.

Acknowledgements. I thank Andrew Barbour for many helpful comments and suggestions, and a referee for many careful and constructive suggestions.

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